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Tensor operators and Wigner–Eckart theorem for the quantum superalgebra $U_q[osp(1\,|\,2)]$

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Abstract

Tensor operators in graded representations of Z_2-graded Hopf algebras are defined and their elementary properties are derived. The Wigner–Eckart theorem for irreducible tensor operators for $U_q[osp(1\,|\,2)]$ is proven. Examples of tensor operators in the irreducible representation space of Hopf algebra $U_q[osp(1\,|\,2)]$ are considered. The reduced matrix elements for the irreducible tensor operators are calculated. A construction of some elements of the centre of $U_q[osp(1\,|\,2)]$ is given.

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1. Introduction

This paper is a continuation of the study of the properties of irreducible representations (the so-called Racah–Wigner calculus) of the quantum superalgebra $U_q[osp(1\,|\,2)]$. In previous papers [1–3] it was shown that it is possible to construct Racah–Wigner calculus for this quantum superalgebra in a completely similar way as in the classical Lie algebra $su(2)$ [4] and the quantum algebra $U_q(su(2))$ [5–7]. It is quite remarkable that all topics that are relevant for the Racah–Wigner calculus for $su(2)$ or $U_q(su(2))$ have their direct super-analogue in the representation theory quantum superalgebra $U_q[osp(1\,|\,2)]$.

An important part of the classical Racah–Wigner calculus is the definition and properties of tensor operators in the representation spaces. The concept of tensor operators is very important in applications of symmetry techniques (Lie groups and algebras) in theoretical physics. The irreducible tensor operators for the Lie group of space rotations were first introduced by Wigner [8]. An equivalent definition of tensor operators for the corresponding Lie algebra was given by Racah [9]. These tensor operators play a very important role in the theory of angular momentum in quantum physics.

The importance of tensor operators in the representation theory of the Lie groups and algebras leads us naturally to investigate the concept of tensor operators for quantum groups and algebras as well as for the quantum superalgebras. The classical Wigner–Racah definition of the irreducible tensor operator has been extended to the quantum Lie algebras in [6, 7, 10] and the Wigner–Eckart theorem has been proved in a similar way as in classical undeformed symmetry structures. In [11, 12] a new, more general definition of tensor operators for arbitrary Hopf algebra has been proposed. According to these definitions tensor operators are homomorphisms of some Hopf algebra representations. The new general definitions, on one hand, are equivalent to the classical Wigner–Racah definitions if the corresponding Hopf algebra is $su(2)$ or $U_q(su(2))$, on the other hand they allow us to deduce easier general properties of tensor operators from basic properties of Hopf algebra representations. The Wigner–Eckart theorem for irreducible tensor operators for Hopf algebras has been proved in paper [13], where a more general version of the definition from paper [12] has been used.

In this paper we define tensor operators for $\mathbb{Z}_2$-graded Hopf algebras in a similar way as in papers [12, 13]. We study the basic properties of the linear operators acting in the graded irreducible representation spaces of the quantum superalgebra $U_q[osp(1\,|\,2)]$. In particular we prove the Schur lemma for $U_q[osp(1\,|\,2)]$. Next we formulate and prove the Wigner–Eckart theorem for irreducible tensor operators of the quantum superalgebra $U_q[osp(1\,|\,2)]$. The proof is based on the properties of $U_q[osp(1\,|\,2)]$ representations, in particular conclusions from the Schur lemma play an important role in it. It is remarkable that the Wigner–Eckart theorem for $U_q[osp(1\,|\,2)]$ has exactly the same form as in the classical case $su(2)$ or $U_q(su(2))$ i.e. the matrix elements of components of the irreducible tensor operator for $U_q[osp(1\,|\,2)]$ are proportional to Clebsch–Gordan coefficients and the proportionality coefficient (reduced matrix element) has the same properties as that in the case of $su(2)$ or $U_q(su(2))$. Using properties of representations of graded Hopf algebras we construct two classes of tensor operators for $U_q[osp(1\,|\,2)]$. In the first class tensor operators act in the adjoint and regular representations of $U_q[osp(1\,|\,2)]$. The second class of tensor operators consists of the irreducible tensor operators acting in the irreducible representation spaces of $U_q[osp(1\,|\,2)]$. As an application of the Wigner–Eckart theorem we calculate the reduced matrix elements for the irreducible tensor operators. Finally we give a method of constructing the elements of the centre of $U_q[osp(1\,|\,2)]$, based on the properties of the tensor product of irreducible representations.

This paper has the following structure. In section 2 we give a review of basic definitions and properties of graded representations, we define tensor operators for $\mathbb{Z}_2$-graded Hopf algebra and we give some examples of tensor operators. In section 3 we review basic properties of grade star representations of $U_q[osp(1\,|\,2)]$. Using these properties we prove the Schur lemma, next we formulate and prove the Wigner–Eckart theorem for the quantum superalgebra $U_q[osp(1\,|\,2)]$. In section 4 we consider examples of tensor operators for $U_q[osp(1\,|\,2)]$, calculate the reduced matrix element for the irreducible ones and give a construction of some elements of the centre of $U_q[osp(1\,|\,2)]$.

2. Tensor operators for $\mathbb{Z}_2$-graded Hopf algebras

We begin by recalling the definition of the $\mathbb{Z}_2$-graded Hopf algebra.

Definition 1. *A $\mathbb{Z}_2$-graded Hopf algebra is a vector space A over complex field* $\mathbf{C}$ *such that $A = \oplus_{\alpha\in\mathbb{Z}_2} A_\alpha$. The elements a of A_α are said to be homogeneous of degree α ($\alpha = 0 \leftrightarrow$ even, $\alpha = 1 \leftrightarrow$ odd) and their degree will be denoted* $\deg(a) \equiv |a| \in \mathbb{Z}_2$. *We assume that the* *unit* $\mathbf{1}$ *of a graded algebra belongs to* A_0. *In the following all Greek indices will belong to* Z_2. *Further we have in* A

(1) an associative multiplication, $m : A \otimes A \to A, m(A_\alpha \otimes A_\beta) \subset A_{\alpha+\beta}, m(a \otimes b) = ab, a, b \in A$,

$$m \circ (\mathrm{id}_A \otimes m) = m \circ (m \otimes \mathrm{id}_A)$$

(2) a coassociative comultiplication, $\Delta : A \to A \otimes A, |a \otimes b| = |a| + |b|, \Delta : A_\alpha \subset \oplus_{\beta+\gamma=\alpha} A_\beta \otimes A_\gamma, \Delta(a) = \sum_i a_i^{(1)} \otimes b_i^{(2)}, a \in A$,

$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta$$

(3) a counit, $\varepsilon : A \to \mathbf{C}$,

$$(\mathrm{id}_A \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A$$

and we have $\varepsilon(A_1) = 0$

(4) an antipode $S : A \to A, S(A_\alpha) \subset A_\alpha$

$$m \circ (\mathrm{id}_A \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}_A) \circ \Delta = \mathrm{i} \circ \varepsilon$$

such that the mappings Δ *and* ε *are algebra homomorphisms* Z_2*-graded algebras and in particular the multiplication in* $A \otimes A$ *is given by*

$$(a \otimes b)(c \otimes d) = (-1)^{|c||b|}(ac \otimes bd).$$

One can show that the antipode S is always an anti-homomorphism of the algebra and of the coalgebra,

$$S(ab) = (-1)^{|a||b|} S(a)S(b), \qquad (S \otimes S) \circ \Delta = \tau \circ \Delta \circ S.$$

where the map $\tau : A \otimes A \to A \otimes A$ is given by

$$\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

We will need later on the following identity

$$\sum_{i,j} \big(a_i^{(1)}\big)_j^{(1)} \otimes S\big(a_i^{(1)}\big)_j^{(2)} a_i^{(2)} = a \otimes \mathbf{1} \tag{2.1}$$

where $a \in A$. This identity follows from coassociativity of the coproduct Δ.

The simplest example of Z_2-graded Hopf algebra is the quantum superalgebra $U_q[osp(1\,|\,2)]$. The quantum superalgebra $U_q[osp(1\,|\,2)]$ is Z_2-graded algebra with unit $\mathbf{1}$ and generated by three elements: H ($\deg(H) = 0$) and $v_\pm$ ($\deg(v_\pm) = 1$) with the following (anti)commutation relations

$$[H, v_\pm] = \pm\frac{1}{2} v_\pm; \qquad [v_+, v_-]_+ = -\frac{sh(\eta H)}{sh(2\eta)} \tag{2.2}$$

where the parameter η is real and we set $q = \mathrm{e}^{-\frac{\eta}{2}}$. The following formulae for coproduct Δ, antipode S and the counit ε define on $U_q[osp(1\,|\,2)]$ the structure of Z_2-graded Hopf algebra

$$\Delta(H) = H \otimes \mathbf{1} + \mathbf{1} \otimes H; \qquad \Delta(v_\pm) = v_\pm \otimes q^H + q^{-H} \otimes v_\pm,$$

$$\varepsilon(H) = \varepsilon(v_\pm) = 0, \qquad \varepsilon(\mathbf{1}) = 1$$

and the antipode is defined by

$$S(H) = -H; \qquad S(v_\pm) = -q^{\pm\frac{1}{2}} v_\pm.$$

As of Z_2-graded Hopf algebra $U_q[osp(1\,|\,2)]$ has the form $U_q[osp(1\,|\,2)] = \oplus_{\alpha\in Z_2} (U_q[osp(1\,|\,2)])_\alpha$.

For any Z_2-graded Hopf algebra A, one can define the adjoint action ad of A on itself in the following way

$$\mathrm{ad}_a(b) = \sum_i (-1)^{|a_i^{(2)}||b|} a_i^{(1)} b S\big(a_i^{(2)}\big)$$

for any $a, b \in A$. Using this action we define the subset of invariant elements of A

$$A_\varepsilon = \{b \in A : \mathrm{ad}_a(b) = \varepsilon(a)b, \forall a \in A\}.$$

We will need later on the following proposition which characterizes the invariant elements of Z_2-graded Hopf algebra A.

Proposition 1. *An element $b \in A$ is ad-invariant if and only if it belongs to the centre $Z(A)$ of A i.e. we have for any $a \in A$*

$$\mathrm{ad}_a(b) = \sum_i (-1)^{|a_i^{(2)}||b|} \big(a_i^{(1)}\big) b S\big(a_i^{(2)}\big) = \varepsilon(a)b \Leftrightarrow ab = (-1)^{|a||b|} ba$$

or equivalently we have $A_\varepsilon = Z(A)$.

Proof. First we prove ($\Rightarrow$). If $b \in Z(A)$ then we have for any $a \in A$

$$\mathrm{ad}_a(b) = \sum_i (-1)^{|a_i^{(2)}||b|} \big(a_i^{(1)}\big) b S\big(a_i^{(2)}\big) = \sum_i \big(a_i^{(1)}\big)\big(S\big(a_i^{(2)}\big)\big) b = \varepsilon(a)b$$

The proof of the converse ($\Leftarrow$) is more difficult. Now we assume that $b \in A_\varepsilon$ i.e. for any $a \in A$

$$\mathrm{ad}_a(b) = \sum_i (-1)^{|a_i^{(2)}||b|} \big(a_i^{(1)}\big) b S\big(a_i^{(2)}\big) = \varepsilon(a) f \tag{2.3}$$

and we have to prove that from this it follows

$$ba = (-1)^{|a||b|} ab \tag{2.4}$$

First let us observe that from $\varepsilon(A_1) = 0$ we have for any $a \in A$

$$\varepsilon(a) = (-1)^{k|a|} \varepsilon(a) \tag{2.5}$$

where k is an arbitrary number. We have also from definition 1 for any i, j appearing in the coproduct $\Delta(a)$

$$\big|a_i^{(1)}\big| = \big|\big(a_i^{(1)}\big)_j^{(1)}\big| + \big|\big(a_i^{(1)}\big)_j^{(2)}\big|. \tag{2.6}$$

We start from the LHS of equation (2.4)

$$ba = \sum_i b\big[\varepsilon\big(a_i^{(1)}\big) a_i^{(2)}\big] = \sum_i (-1)^{|a_i^{(1)}||b|} \varepsilon\big(a_i^{(1)}\big) b a_i^{(2)}$$

where we have used equation (2.5). Now we use equation (2.3) for $a = a_i^{(1)}$ and get

$$\begin{aligned} ba &= \sum_{ij} \left\{ (-1)^{|a_i^{(1)}||b|} (-1)^{|(a_i^{(1)})_j^{(2)}||b|} \big[\big(a_i^{(1)}\big)_j^{(1)}\big] b \big[S\big(a_i^{(1)}\big)_j^{(2)}\big] \right\} a_i^{(2)} \\ &= \sum_{ij} (-1)^{|(a_i^{(1)})_j^{(1)}||b|} \big[\big(a_i^{(1)}\big)_j^{(1)}\big] b \big[S\big(a_i^{(1)}\big)_j^{(2)} a_i^{(2)}\big]. \end{aligned}$$

In the last equation we have used equation (2.6). Now we will prove that

$$\sum_{ij} (-1)^{|(a_i^{(1)})_j^{(1)}||b|} \big[\big(a_i^{(1)}\big)_j^{(1)}\big] b \big[S\big(a_i^{(1)}\big)_j^{(2)} a_i^{(2)}\big] = (-1)^{|a||b|} ab$$

From the coassociativity condition for the coproduct Δ we get

$$\sum_{i,j} b \otimes \left(a_i^{(1)}\right)_j^{(1)} \otimes \left(a_i^{(1)}\right)_j^{(2)} \otimes a_i^{(2)} = \sum_{i,j} b \otimes a_i^{(1)} \otimes \left(a_i^{(2)}\right)_j^{(1)} \otimes \left(a_i^{(2)}\right)_j^{(2)}$$

Acting on both sides of the above equation by $(m \circ (m \otimes \mathrm{id}) \circ (m \otimes \mathrm{id} \otimes \mathrm{id})) \circ (\tau \otimes S \otimes \mathrm{id})$ we get

$$\sum_{i,j} (-1)^{|(a_i^{(1)})_j^{(1)})||b|} \left(a_i^{(1)}\right)_j^{(1)} b S\left(a_i^{(1)}\right)_j^{(2)} a_i^{(2)} = (-1)^{|a||b|} \sum_{i,j} (-1)^{|a_i^{(2)}||b|} a_i^{(1)} b S\left(a_i^{(2)}\right)_j^{(1)} \left(a_i^{(2)}\right)_j^{(2)}$$

$$= (-1)^{|a||b|} \sum_i (-1)^{|a_i^{(2)}||b|} a_i^{(1)} b \varepsilon\left(a_i^{(2)}\right) = (-1)^{|a||\mathrm{b}|} ab.$$

□

In the following we will consider the representations of Z_2-graded Hopf algebra $U_q[osp(1\,|\,2)]$ in the Z_2-graded linear spaces therefore we recall here some basic properties of the graded representations [17]. A vector space V over complex field $\mathbf{C}$ is called a Z_2-graded linear space or simply graded space if $V = \oplus_{\alpha \in Z_2} V_\alpha$. The elements v of V_α are said to be homogeneous of degree α ($\alpha = 0 \leftrightarrow$ even, $\alpha = 1 \leftrightarrow$ odd) and their degree will be denoted similarly as in the case of graded algebras $\deg(v) \equiv |v| \in Z_2$. Consider now two graded vector spaces V, W and a linear mapping $f \in \mathrm{Hom}(V, W)$. The mapping f is said to be homogeneous of degree $\beta \in Z_2$ if

$$f(V_\alpha) \subset W_{\alpha+\beta}.$$

where $\alpha \in Z_2$. So we get a gradation in linear space $\mathrm{Hom}(V, W)$

$$\mathrm{Hom}(V, W)_\beta = \{f \in \mathrm{Hom}(V, W) : f(V_\alpha) \subset W_{\alpha+\beta}\}.$$

and

$$\mathrm{Hom}(V, W) = \mathrm{Hom}(V, W)_0 \oplus \mathrm{Hom}(V, W)_1$$

For a given Z_2-graded Hopf algebra A, a graded representation of A is defined in the following way:

Definition 2. *A graded representation of Z_2-graded Hopf algebra A in Z_2-graded linear space V is an even homomorphism $\rho : A \to \mathrm{Hom}(V, V)$, i.e. $\rho \in \mathrm{Hom}(A, \mathrm{Hom}(V, V))$. The pair (V, ρ) is called a graded representation of Hopf algebra A. The representation (V, ρ) is irreducible if there is no proper subspace $V' \subset V$ which is invariant under action of the Hopf algebra A via map ρ.*

Let us recall some examples of Z_2-graded Hopf algebra representations.

Example 1. A Z_2-graded Hopf algebra A is itself a graded representation space for the adjoint action $\rho(a) \equiv \mathrm{ad}_a$

$$\mathrm{ad}_a(b) = \sum_i (-1)^{|a_i^{(2)}||b|} a_i^{(1)} b S\left(a_i^{(2)}\right)$$

for $a, b \in A$. This representation is denoted by $(A, \mathrm{ad}) \equiv A_{ad}$.

Example 2. A Z_2-graded Hopf algebra A is also a graded representation space for a left regular action L of A

$$L(a) \cdot b = m(a \otimes b) = ab$$

for any $a, b \in A$. A left regular representation is denoted by $(A, L) \equiv A_L$.

Example 3. Let (V, π), and (W, ρ) be graded modules of $\mathbb{Z}_2$-graded Hopf algebra A. The linear space $\mathrm{Hom}(V, W)$ is a graded A-module $(\mathrm{Hom}(V, W), \delta)$ with the action of A on $f \in \mathrm{Hom}(V, W)$ defined as follows:

$$\delta(a)(f) = \sum_i (-1)^{|a_i^{(2)}||f|} \rho\big(a_i^{(1)}\big) \circ f \circ \pi\big(S\big(a_i^{(2)}\big)\big).$$

Example 4. The tensor product $V \otimes W$ of two graded representation spaces of representations (V, π), and (W, ρ) is a graded representation space where the action $\delta^{\otimes}$ of A is the following:

$$\delta^{\otimes}(a)(v \otimes w) = \sum_i (-1)^{|a_i^{(2)}||v|} \pi\big(a_i^{(1)}\big) v \otimes \rho\big(S\big(a_i^{(2)}\big)\big) w.$$

for any $v \in V, w \in W$ and where $|v \otimes w| = |v| + |w|$. This yields the representation $(W \otimes V, (\rho \otimes \pi) \circ \Delta)$.

The last example is the following:

Example 5. The counit map ε of A equips any graded vector space V with a trivial representation $\rho = \varepsilon$ structure where

$$av = \varepsilon(a)v$$

where $v \in V$ and $a \in A$. In particular any one-dimensional representation (which is not a zero representation) is equivalent to a trivial representation.

The concept of trivial action of the $\mathbb{Z}_2$-graded Hopf algebra A on vectors of representation space can be applied to any representation of A.

Definition 3. *For any representation (V, ρ) of Hopf algebra A we define the subspace of invariant vectors*

$$V_\varepsilon = \{v \in V : \rho(a) \cdot v = \varepsilon(a)v, \forall a \in A\}.$$

The next important mathematical tool which we are going to use later on is a graded intertwiner of representations so let us recall its definition.

Definition 4. *Let (V, ρ) and (W, σ) be representations of the $\mathbb{Z}_2$-graded Hopf algebra A. A linear map $f \in \mathrm{Hom}(V, W)$ is a graded intertwiner of representations (V, π) and (W, ρ) if*

$$f \circ \pi(a) = (-1)^{|a||f|} \rho(a) \circ f.$$

for any $a \in A$. The space of the graded intertwiners will be denoted $I_A(V, W)$. An even intertwiner is a homomorphism of representations so the subspace $(I_A(V, W))_0 \equiv \mathrm{Hom}_A(V, W)$ is a space of homomorphisms.

We give two examples of homomorphisms of representations of A, which will be important in the following.

Example 6. The $\mathbb{Z}_2$-graded Hopf algebra A with the adjoint action $\mathrm{ad}_a, a \in A$ form the adjoint representation (A, ad_a). On the other hand we have the representation $(\mathrm{Hom}(V, V), \delta)$ of A from example 3. The representation $\rho : A \to \mathrm{Hom}(V, V)$ from definition 2 is a homomorphism of the Hopf algebra representations.

Example 7. A left regular action L given in example 2 is a homomorphism of representations A_{ad} and $(\mathrm{Hom}(A_L, A_L), \delta)$, i.e. $L \in \mathrm{Hom}_A(A_{ad}, \mathrm{Hom}(A_L, A_L))$. In fact we have for any $a, b \in A$

$$L(\mathrm{ad}_a(b)) = \sum_i (-1)^{|a_i^{(2)}||b|} L\big(a_i^{(1)} b S\big(a_i^{(2)}\big)\big) = \sum_i (-1)^{|a_i^{(2)}||b|} L\big(a_i^{(1)}\big) L\big(b\big) L\big(S(a_i^{(2)})\big)$$

or equivalently

$$L \circ \mathrm{ad}_a = \delta(a) \circ L.$$

where $|L| = 0$ because L is a representation.

Now we are in a position to define tensor operators for $\mathbb{Z}_2$-graded Hopf algebras. Following the idea of the definition of tensor operators for Hopf algebras given in [12] we define tensor operators for $\mathbb{Z}_2$-graded Hopf algebras in the following way:

Definition 5. *Let (V, π), (W, ρ) and (U, σ) be graded representations of the $\mathbb{Z}_2$-graded Hopf algebra A and let $T \in \mathrm{Hom}(V, \mathrm{Hom}(W, U))$ then T is a tensor operator of type V in W if $T \in I_A(V, \mathrm{Hom}(W, U))$. In other words tensor operator T is a graded intertwiner of representations (V, π) and $(\mathrm{Hom}(W, U), \delta)$ and it satisfies*

$$T \circ \pi(a) = (-1)^{|a||T|} \delta(a) \circ T. \tag{2.7}$$

Let vectors $\{e_l\}_{l \in I \subset N}$ be a basis of the representation space V, then the linear operators $T(e_l) \equiv T_l \in \mathrm{Hom}(W, U)$ will be called the components of the tensor operator T. If $\dim V < \infty$ then the components T_l of T satisfy

$$\pi(a)_{jl} T_j = (-1)^{|a||T|} \sum_i (-1)^{|a_i^{(2)}||T_l|} \sigma\big(a_i^{(1)}\big) \circ T_l \circ \rho\big(S\big(a_i^{(2)}\big)\big) \tag{2.8}$$

where $\pi(a)_{jl}$ is a matrix of $\pi(a)$. If all the representations (V, π), (W, ρ) and (U, σ) are irreducible then the tensor operator T is called irreducible.

Let us write the defining equation (2.8) for the components T_l of T when $A = U_q[osp(1\,|\,2)]$ and $a = v_\pm, H$

$$\pi(v_+)_{jl} T_j = (-1)^{|v_+||T|} (\sigma(v_+) \circ T_l \circ \rho(q^{-H}) - (-1)^{|v_+||T_l|} q^{\frac{1}{2}} \sigma(q^{-H}) \circ T_l \circ \rho(v_+)) \tag{2.9}$$

$$\pi(v_-)_{jl} T_j = (-1)^{|v_-||T|} (\sigma(v_-) \circ T_l \circ \rho(q^{-H}) - (-1)^{|v_-||T_l|} q^{-\frac{1}{2}} \sigma(q^{-H}) \circ T_l \circ \rho(v_-)) \tag{2.10}$$

$$\pi(H)_{jl} T_j = \sigma(H) \circ T_l - T_l \circ \rho(H) \tag{2.11}$$

Thus the above definition of tensor operator, although it seems to be abstract in the case of the simplest quantum superalgebra $U_q[osp(1\,|\,2)]$ which is a superanalogue of the quantum algebra $U_q[su(2))]$, gives very similar defining formulae for generating elements as in the case of $U_q[su(2))]$ [6, 7, 10].

Let us give some important examples of tensor operators.

Example 8. Example 6 shows that the representation ρ from definition 2 is itself a tensor operator because $\rho \in \mathrm{Hom}_A(A, \mathrm{Hom}(W \otimes W))$.

Example 9. The left regular action L of A on itself as defined in example 2 is a tensor operator because $L \in \mathrm{Hom}_A(A_{ad}, \mathrm{Hom}(A_L, A_L))$ (example 7).

Before formulating a lemma which will be used later on we introduce useful notation. If $f \in \mathrm{Hom}(V, W)$ where (V, π) and (W, ρ) are representations of the Hopf algebra a, then we define

$$\pi_f(a) \equiv f \circ \pi(a) : V \to W \tag{2.12}$$

and the linear mapping $m^{\pi}_{\rho} : \mathrm{Hom}(W) \otimes \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ is defined in the following way

$$m^{\pi}_{\rho}(\rho(a) \otimes \pi_f(b)) = (-1)^{|a||f|}\left(\left(m^{\pi}_{\rho} \circ (\rho \otimes \pi_f)\right)\right) \cdot (a \otimes b) \equiv \rho(a) \circ \pi_f(b). \tag{2.13}$$

Lemma 1. *Assume that*

(1) (V, π), $W, \rho)$, (U, σ) *and* $(\mathrm{Hom}(W, U), \delta)$ *are representations of the* $\mathbb{Z}_2$*-graded Hopf algebra* A,
(2) $T \in I_A(V, \mathrm{Hom}(W, U))$, i.e. $\forall a \in A\ T \circ \pi(a) = (-1)^{|a||T|}\delta(a) \circ T$,
(3) $\check{T} \in \mathrm{Hom}(V \otimes W, U))$ *and* $\check{T}(v \otimes w) \equiv T(v) \cdot w\ \forall v \in V, w \in W$.

Then

(a) $\check{T} \in I_A(V \otimes W, U))$, i.e. $\forall a \in A\ \check{T} \circ [(\pi \otimes \rho)\Delta(a)] = (-1)^{|a||\check{T}|}\sigma(a) \circ \check{T}$.
(b) $|T| = |\check{T}|$

Proof. Let us prove (a). The action δ of representation $(\mathrm{Hom}(W, U), \delta)$ is given in example 3. We rewrite condition (2) for T in the form

$$T[\pi(a) \cdot v] \cdot w = (-1)^{|a||T|}\left\{\sum_i (-1)^{|a_i^{(2)}||T(v)|}\sigma\left(a_i^{(1)}\right) \circ T(v) \circ \rho\left(S\left(a_i^{(2)}\right)\right)\right\} \cdot w \tag{2.14}$$

for any $a \in A$, $v \in V$, $w \in W$. We have to prove that from this follows condition (a) for $\check{T}$ which can be written as follows

$$\sum_i (-1)^{|a_i^{(2)}||v|}T\left[\pi\left(a_i^1\right) \cdot v\right] \cdot \left(\rho\left(a_i^{(2)}\right) \cdot w\right) = (-1)^{|a||T|}\sigma(a)[T(v) \cdot w] \tag{2.15}$$

for any $a \in A$, $v \in V$, $w \in W$. Applying condition (2.14) for $a = a_i^1$ to the LHS of the above equation we get

$$\sum_i (-1)^{|a_i^{(2)}||v|}T\left[\pi\left(a_i^1\right) \cdot v\right] \cdot \left(\rho\left(a_i^{(2)}\right) \cdot w\right) = \sum_{ij} (-1)^{|a_i^{(2)}||v|}(-1)^{|a_i^1||T|}$$
$$\times (-1)^{|(a_i^{(1)})_j^{(2)}||T(v)|}\sigma\left[\left(a_i^{(1)}\right)_j^{(1)}\right] \circ T(v) \circ \rho\left[S\left(a_i^{(1)}\right)_j^{(2)}\left(a_i^{(2)}\right)\right] \cdot w.$$

In the notation (2.12), (2.13) it takes the form

$$\sum_i (-1)^{|a_i^{(2)}||v|}T\left[\pi\left(a_i^1\right) \cdot v\right] \cdot \left(\rho\left(a_i^{(2)}\right) \cdot w\right) = \sum_{ij} (-1)^{|a_i^{(2)}||v|}(-1)^{|a_i^1||T|}(-1)^{|(a_i^{(1)})_j^{(2)}||T(v)|}$$
$$\times (-1)^{|(a_i^{(1)})_j^{(1)}||T(v)|}\left\{m^{\rho}_{\sigma} \circ (\sigma \otimes \rho_{T(v)}) \cdot \left(\left(a_i^{(1)}\right)_j^{(1)} \otimes S(a_i^{(1)})_j^{(2)} a_i^{(2)}\right)\right\} \cdot w.$$

Simpifying the phase and using the identity (2.1) we get

$$\sum_i (-1)^{|a_i^{(2)}||v|}T\left[\pi\left(a_i^1\right) \cdot v\right] \cdot \left(\rho\left(a_i^{(2)}\right) \cdot w\right) = (-1)^{|a||v|}\left\{m^{\rho}_{\sigma} \circ (\sigma \otimes \rho_{T(v)}) \cdot (a \otimes \mathbf{1})\right\} \cdot w$$
$$= (-1)^{|a||v|}(-1)^{|a||T(v)|}\sigma(a)[T(v) \cdot w]$$
$$= (-1)^{|a||v| + |a||T(v)|}\sigma(a)[T(v) \cdot w]$$

which is the RHS of equation (2.15). Statement (b) can be proved considering the degrees of the values of T and $\check{T}$ on homogeneous arguments. $\square$

3. Wigner–Eckart theorem for the quantum superalgebra $U_q[osp(1\,|\,2)]$

In this section we will consider the quantum superalgebra $U_q[osp(1\,|\,2)]$ and its graded representations. A representation of the quantum superalgebra $U_q[osp(1\,|\,2)]$ in the graded linear space V will be denoted by π

$$\pi : U_q[osp(1\,|\,2)] \to \mathrm{Hom}(V, V).$$

The finite dimensional irreducible representations of $U_q[osp(1\,|\,2)]$ were first studied in [15]. They have the same structure as in case of the nondeformed superalgebra $osp(1\,|\,2)$ and for this superalgebra every finite dimensional irreducible representation is equivalent to a grade star representation [16]. It has been shown in [2] that any finite dimensional grade star representation of $U_q[osp(1\,|\,2)]$ is characterized by four parameters: the highest weight l (a non-negative integer), the parity $\lambda = 0, 1$ of the highest weight vector in the representation space and by $\varphi, \psi = 0, 1$, the signature parameters of the Hermitian in the representation space V. The parity λ and the signature φ define the class $\epsilon = 0, 1$ of the grade star representation by

$$\epsilon = \lambda + \varphi + 1, \mathrm{mod}(2).$$

For simplicity we will write $(V^l(\lambda), \pi^l)$ instead of $\left(V^l(\lambda), \pi^{l\epsilon}_{\varphi\psi}\right)$ The representation space $V^l(\lambda)$ is a graded vector space of dimension $2l+1$ with basis $e^l_m(\lambda)$ where $-l \leqslant m \leqslant l$. The parity of the basis vectors $e^l_m(\lambda)$ is determined by values of l, m and λ

$$\left|e^l_m(\lambda)\right| = l - m + \lambda \,\mathrm{mod}(2).$$

The vectors $e^l_m(\lambda)$ are pseudo-orthogonal with respect to the Hermitian form in V and their normalization is determined by the signature parameters φ, ψ

$$\left(e^l_m(\lambda), e^{l'}_{m'}(\lambda)\right) = (-1)^{\varphi(l-m)+\psi}\delta_{mm'},$$

where (,) denotes the Hermitian form in the representation space $V^l(\lambda)$. The operators $\pi^l(v_\pm)$ and $\pi^l(H)$ act on the basis $e^l_m(\lambda)$ in the following way

$$\pi^l(v_+)\cdot e^l_m = (-1)^{(l-m)}([l-m][l+m+1]\gamma)^{\frac{1}{2}} e^l_{m+1} \tag{3.1}$$

$$\pi^l(v_-)\cdot e^l_m = ([l+m][l-m+1]\gamma)^{\frac{1}{2}} e^l_{m-1} \tag{3.2}$$

$$\pi^l(H)\cdot e^l_m = \frac{m}{2} e^l_m \tag{3.3}$$

where $[n] = \frac{q^{-\frac{n}{2}} - (-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}$ and $\gamma = \frac{ch(\frac{\eta}{4})}{sh(2\eta)}$. Note that the action of the operators $\pi^l(v_\pm)$ and $\pi^l(H)$ does not depend on the parameters λ, φ, ψ.

The tensor product of two irreducible representations $(V^{l_1}(\lambda_1), \pi^{l_1})$ and $(V^{l_2}(\lambda_2), \pi^{l_2})$ is completely and simply reducible, i.e. we have

$$V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2) = \oplus^{l_1+l_2}_{l=|l_1-l_2|} V^l(\lambda).$$

By definition the Clebsch–Gordan coefficients (C-Gc) $(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2\,|\,lm\lambda)_q$ relate the standard basis $e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)$ of tensor product $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ with the reduced basis $e^l_m(l_1, l_2, \lambda)$ in the following way

$$e^l_m(l_1, l_2, \lambda) = \sum_{m_1 m_2} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2\,|\,lm\lambda)_q\, e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)$$

or equivalently

$$(-1)^{(l_1-m_1)(l_2-m_2)} e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) = \sum_{lm} (-1)^{(l-m)L} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2\,|\,lm\lambda)_q\, e^l_m(l_1, l_2, \lambda)$$

where $m_1 + m_2 = m$, $L = l_1 + l_2 + l$ and l is an integer satisfying the condition

$$|l_1 - l_2| \leqslant l \leqslant l_1 + l_2.$$

In the following, in order to get the Wigner–Eckart theorem in a conventional form we will use modified C-Gc $[l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \mid lm\lambda]_q$ which are related to $(l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \mid lm\lambda)_q$ by

$$[l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \mid lm\lambda]_q = (-1)^{(l_1 - m_1)(l_2 - m_2)} (-1)^{(l-m)L} (l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \mid lm\lambda)_q.$$

In terms of the modified C-Gc the relation between the standard and the reduced basis in $V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)$ looks like

$$(-1)^{(l-m)L} e^l_m(l_1, l_2, \lambda) = \sum_{m_1 m_2} (-1)^{(l_1 - m_1)(l_2 - m_2)} [l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \mid lm\lambda]_q e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)$$

or equivalently

$$e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) = \sum_{lm} [l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \mid lm\lambda]_q e^l_m(l_1, l_2, \lambda). \tag{3.4}$$

We have also for any l, m in this decomposition

$$\left|e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)\right| = \left|e^l_m(l_1, l_2, \lambda)\right|. \tag{3.5}$$

In the classical theory of Racah–Wigner calculus, a very important role is played by the C-Gc $(jm, jn \mid 00)$, which defines an invariant metric. In the case of the quantum superalgebra $U_q[osp(1 \mid 2)]$, the corresponding coefficient also defines an invariant metric. It has the form

$$C^l_{mn}(\lambda) = \sqrt{[2l+1]}(lm\lambda, \ln\lambda \mid 00)_q = (-1)^{(l-m)\lambda} (-1)^{(l-m)(l-m-1)/2} q^{m/2} \delta m, -n. \tag{3.6}$$

For more details on the irreducible grade star representations and properties of C-Gc, see [2].

In the case of the irreducible finite dimensional representations of the quantum superalgebra $U_q[osp(1 \mid 2)]$ the Schur lemma has the following form

Lemma 2. *Let $(V^{l_1}(\lambda_1), \pi^{l_1})$ and $(V^{l_2}(\lambda_2), \pi^{l_2})$ be irreducible finite dimensional representations of $U_q[osp(1 \mid 2)]$ and let $f \in I_{U_q[osp(1|2)]}(V^{l_1}(\lambda_1), V^{l_2}(\lambda_2))$, i.e. for any $a \in U_q[osp(1 \mid 2)]$, $x \in V^{l_1}(\lambda_1)$*

$$f(\pi^{l_1}(a) \cdot x) = (-1)^{|f||a|} \pi^{l_2}(a) f(x), \tag{3.7}$$

then $f = \alpha\, \mathrm{id}_{V^{l_1}(\lambda_1)} (\alpha \in \mathrm{R})$ if $l_1 = l_2$ and $\lambda_1 = \lambda_2$, or $f = 0$ if $l_1 \neq l_2$ or $\lambda_1 \neq \lambda_2$.

Proof. Let us consider the properties of the vector

$$y = f\left(e^{l_1}_{l_1}(\lambda_1)\right) \in V^{l_2}(\lambda_2).$$

Using equation (3.7) we get

$$\pi^{l_2}(H) \cdot y = \frac{l_1}{2} y; \pi^{l_2}(v_+) \cdot y = 0$$

so either $y \in V^{l_2}(\lambda_2)$ is the highest weight vector of weight l_1 in $V^{l_2}(\lambda_2)$ or $f = 0$, i.e. either $l_1 = l_2$ or $f = 0$. Assume that $l_1 = l_2$ and λ_1, λ_2 arbitrary. Then from the above it follows that we have

$$f\left(e^{l_1}_{m_1}(\lambda_1)\right) = \alpha e^{l_1}_{m_1}(\lambda_2) \tag{3.8}$$

and $|f| = 1$ if $\lambda_1 + \lambda_2 = 1$ or $|f| = 0$ if $\lambda_1 + \lambda_2 = 0$ mod(2). Acting on both sides of the above equation by $T^{l_1}(v_+)$ we get

$$(-1)^{(l_1 - m_1)}([l_1 - m_1][l_1 + m_1 + 1]\gamma)^{\frac{1}{2}} e^l_{m_1+1}(\lambda_2)$$
$$= (-1)^{|f|}(-1)^{(l_1 - m_1)}([l_1 - m_1][l_1 + m_1 + 1]\gamma)^{\frac{1}{2}} e^l_{m_1+1}(\lambda_2)$$

so $f = 0$ if $\lambda_1 + \lambda_2 = 1$. □

We will need later on the following proposition which is a consequence of the Schur lemma

Proposition 2. *Let* $(V^{l_1}(\lambda_1), \pi^{l_1})$, $(V^{l_2}(\lambda_2), \pi^{l_2})$ *and* $(V^{l_3}(\lambda_3), \pi^{l_3})$ *be irreducible finite dimensional representations of* $U_q[osp(1\,|\,2)]$ *with bases respectively* $\{e^{l_2}_{m_1}(\lambda_1)\}$, $\{e^{l_1}_{m_2}(\lambda_2)\}$, $\{e^{l_3}_{m_3}(\lambda_3)\}$ *and let* $f \in I_{U_q[osp(1\,|\,2)]}(V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)), V^{l_3}(\lambda_3))$ *where* $|l_1 - l_2| \leqslant l_3 \leqslant l_1 + l_2$. *Then*

$$f\big(e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)\big) = \alpha_{l_3} \sum_{m_3} [l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \,|\, l_3 m_3 \lambda_3]_q e^{l_3}_{m_3}(l_1, l_2, \lambda_3). \tag{3.9}$$

for any $e^{l_i}_{m_i}(\lambda_i) \in V^{l_i}(\lambda_i)$, $i = 1, 2$ *and* $f \in \big(I_{U_q[osp(1\,|\,2)]}(V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)), V^{l_3}(\lambda_3)\big)_0$, *i.e.* f *is an homomorphism.*

Proof. From Clebsch–Gordan decomposition we have

$$e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2) = \sum_{lm} [l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \,|\, l m \lambda]_q e^{l}_{m}(l_1, l_2, \lambda) \tag{3.10}$$

and for any $|l_1 - l_2| \leqslant l \leqslant l_1 + l_2$ the linear mapping $f_l = f\,|_{V^l(\lambda)} : V^l(\lambda) \to V^{l_3}(\lambda_3)$ is an intertwiner of representations $V^l(\lambda)$ and $V^{l_3}(\lambda_3)$, i.e. $f_l \in I_{U_q[osp(1\,|\,2)]}(V^l(\lambda), V^{l_3}(\lambda_3))$. Therefore we have from the Schur lemma

$$f_l = \alpha_l \, \mathrm{id}_{V^l(\lambda)} \delta_{l l_3} \delta_{\lambda \lambda_3}.$$

Taking into account that $f = \oplus_l f_l$ we get from the Clebsch–Gordan decomposition (3.10) equation (3.9) and it is clear that α_l do not depend on $m_1 m_2, m$. The fact that $f \in \big(I_{U_q[osp(1\,|\,2)]}(V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2)), V^{l_3}(\lambda_3)\big)_0$ follows from relation (3.5). □

Now we can formulate the Wigner–Eckart theorem for irreducible tensor operators for quantum superalgebra $U_q[osp(1\,|\,2)]$.

Theorem 1. *If* $T \in I_{U_q[osp(1\,|\,2)]}(V^{l_1}(\lambda_1), \mathrm{Hom}(V^{l_2}(\lambda_2), V^{l_3}(\lambda_3)))$ *is an irreducible tensor operator. Then*

(1) the matrix elements of its components $T\big(e^{l_1}_{m_1}(\lambda_1)\big)$ *are proportional to the modified Clebsch–Gordan coefficients i.e.*

$$\big[T\big(e^{l_1}_{m_1}(\lambda_1)\big)\big]_{m_3 m_2} = \alpha [l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \,|\, l_3 m_3 \lambda_3]_q$$

where α *is a real number called a reduced matrix element which does not depend on* m_i, $i = 1, 2, 3$.

(2) T *is an even intertwiner, i.e.* $T \in \mathrm{Hom}_{U_q[osp(1\,|\,2)]}(V^{l_1}(\lambda_1), \mathrm{Hom}(V^{l_2}(\lambda_2), V^{l_3}(\lambda_3)))$

Proof. From lemma 1 we know that linear mapping $\check{T} \in \mathrm{Hom}(V^{l_1}(\lambda_1) \otimes V^{l_2}(\lambda_2), V^{l_3}(\lambda_3)))$, $|l_1 - l_2| \leqslant l_3 \leqslant l_1 + l_2$

$$\check{T}\big(e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)\big) = T\big(e^{l_1}_{m_1}(\lambda_1)\big) \cdot e^{l_2}_{m_2}(\lambda_2)$$

is an intertwiner of representations and $|T| = |\check{T}|$. Then from proposition 2 we get

$$\begin{aligned}\check{T}\big(e^{l_1}_{m_1}(\lambda_1) \otimes e^{l_2}_{m_2}(\lambda_2)\big) &= T\big(e^{l_1}_{m_1}(\lambda_1)\big) \cdot e^{l_2}_{m_2}(\lambda_2) \\ &= \alpha \sum_{m_3} [l_1 m_1 \lambda_1, l_2 m_2 \lambda_2 \,|\, l_3 m_3 \lambda_3]_q e^{l_3}_{m_3}(l_1, l_2, \lambda_3)\end{aligned}$$

where α do not depend on m_i, $i = 1, 2, 3$ and $\check{T}$ is even. On the other hand, the matrix of the operator $T\big(e^{l_1}_{m_1}(\lambda_1)\big)$ is defined by the equation

$$T\big(e^{l_1}_{m_1}(\lambda_1)\big)\cdot e^{l_2}_{m_2}(\lambda_2) = \big[T\big(e^{l_1}_{m_1}(\lambda_1)\big)\big]_{m_3 m_2}\cdot e^{l_3}_{m_3}(\lambda_3)$$

Comparing the last two equations we get the statement of the theorem. □

Thus for the quantum superalgebra $U_q[osp(1\,|\,2)]$ the Wigner–Eckart theorem has exactly the same form as in the classical case $su(2)$ and deformed case $U_q[su(2)]$. It is quite a remarkable result because in general all formulae in Racah–Wigner calculus for the quantum superalgebra $U_q[osp(1\,|\,2)]$, have a form similar to corresponding formulae in Racah–Wigner calculus for $su(2)$ and $U_q[su(2)]$, but they differ from the latter by sometimes complicated phases [2, 3]. We have avoided the appearance of the non conventional phase in the Wigner–Eckart theorem using the modified C-Gc.

The irreducible tensor operator T for $U_q[osp(1\,|\,2)]$ is even so we have $\big|T\big(e^l_m(\lambda)\big)\big| = \big|e^l_m(\lambda)\big| = l - m + \lambda \bmod(2)$ and we may introduce notation $T\big(e^l_m(\lambda)\big) \equiv T^l_m(\lambda)$. Let us write the defining relations (2.9)–(2.11) for the components of the irreducible tensor operator $T^l_m(\lambda)$

$$\begin{aligned}
(-1)^{l-m}([l-m][l+m+1]\gamma)^{\frac12}T^l_{m+1}(\lambda) &= \pi^{l_3}(v_+)\circ T^l_m(\lambda)\circ\pi^{l_2}(q^{-H})\\
&\quad - (-1)^{l-m+\lambda}q^{\frac12}\pi^{l_3}(q^{-H})\circ T^l_m(\lambda)\circ\pi^{l_2}(v_+)\\
([l+m][l-m+1]\gamma)^{\frac12}T^l_{m-1}(\lambda) &= \pi^{l_3}(v_-)\circ T^l_m(\lambda)\circ\pi^{l_2}(q^{-H})\\
&\quad - (-1)^{l-m+\lambda}q^{-\frac12}\pi^{l_3}(q^{-H})\circ T^l_m(\lambda)\circ\pi^{l_2}(v_-)\\
\frac{m}{2}T^l_m(\lambda) &= \pi^{l_3}(H)\circ T^l_m(\lambda) - T^l_m(\lambda)\circ\pi^{l_2}(H).
\end{aligned}$$

The above formulae are very similar to defining relations satisfied by the components of the irreducible tensor operator for the Hopf algebra $U_q[su(2)]$ [6, 7, 12, 13]. The difference is only in the phase factor and the definition of the symbol $[n]$. In the limit $q \to 1$, for $l - m = 0 \bmod(2)$ we get

$$\begin{aligned}
\frac12(l-m)^{\frac12}T^l_{m+1}(\lambda) &= \pi^{l_3}(v_+)\circ T^l_m(\lambda) - (-1)^{\lambda}T^l_m(\lambda)\circ\pi^{l_2}(v_+)\\
\frac12(l+m)^{\frac12}T^l_{m-1}(\lambda) &= \pi^{l_3}(v_-)\circ T^l_{m_1}(\lambda) - (-1)^{\lambda}T^l_m(\lambda)\circ\pi^{l_2}(v_-)\\
\frac{m}{2}T^l_m(\lambda) &= \pi^{l_3}(H)\circ T^l_m(\lambda) - T^l_m(\lambda)\circ\pi^{l_2}(H)
\end{aligned}$$

and for $l - m = 1 \bmod(2)$ we have

$$\begin{aligned}
-\frac12(l+m+1)^{\frac12}T^l_{m+1}(\lambda) &= \pi^{l_3}(v_+)\circ T^l_m(\lambda) - (-1)^{\lambda}T^l_m(\lambda)\circ\pi^{l_2}(v_+)\\
\frac12(l-m+1)^{\frac12}T^l_{m-1}(\lambda) &= \pi^{l_3}(v_-)\circ T^l_m(\lambda) - (-1)^{\lambda}T^l_m(\lambda)\circ\pi^{l_2}(v_-)\\
\frac{m}{2}T^l_m(\lambda) &= \pi^{l_3}(H)\circ T^l_m(\lambda_1) - T^l_m(\lambda_1)\circ\pi^{l_2}(H).
\end{aligned}$$

The above equations can be interpreted as defining relations for the components of the irreducible tensor operator for the Lie superalgebra $osp(1\,|\,2)$. It is known that the Lie algebra $sl(2)$ generated by elements $H, L_\pm = \pm 2[v_\pm, v_\pm]_+$ is included in the superalgebra $osp(1\,|\,2)$ and we have

$$[H, L_\pm] = \pm L_\pm; \qquad [l_+, L_-] = 2H.$$

Using the defining relations (2.8) for $a = H, L_\pm$ we get in the limit $q \to 1$ the following equations

$$-\frac{1}{4}\sqrt{(l-m)(l+m+2)}T^l_{m+2}(\lambda_1) = \pi^{l_3}(L_+)\circ T^l_m(\lambda_1) - T^l_m(\lambda_1)\circ\pi^{l_2}(L_+)$$
$$-\frac{1}{4}\sqrt{(l+m)(l-m+2)}T^l_{m-2}(\lambda_1) = \pi^{l_3}(L_-)\circ T^l_m(\lambda_1) - T^l_m(\lambda_1)\circ\pi^{l_2}(L_-)$$
$$\frac{m}{2}T^l_m(\lambda) = \pi^{l_3}(H)\circ T^l_m(\lambda) - T^l_m(\lambda)\circ\pi^{l_2}(H)$$

for $l-m = 0\,\mathrm{mod}(2)$ and

$$-\frac{1}{4}\sqrt{(l-m-1)(l+m+1)}T^l_{m+2}(\lambda) = \pi^{l_3}(L_+)\circ T^l_m(\lambda_1) - T^l_m(\lambda_1)\circ\pi^{l_2}(L_+)$$
$$-\frac{1}{4}\sqrt{(l+m-1)(l-m+1)}T^l_{m-2}(\lambda) = \pi^{l_3}(L_-)\circ T^l_m(\lambda) - T^l_m(\lambda)\circ\pi^{l_2}(L_-)$$
$$\frac{m}{2}T^l_m(\lambda) = \pi^{l_3}(H)\circ T^l_m(\lambda) - T^l_m(\lambda)\circ\pi^{l_2}(H)$$

where $l-m = 1\,\mathrm{mod}(2)$.

These formulae are classical Racah definitions for components of the irreducible tensor operator for the Lie algebra $sl(2)$. Thus in the formal limit $U_q[osp(1\,|\,2)] \to osp(1\,|\,2)$ the set of the components $T^l_m(\lambda)$ of irreducible tensor operator T splits into two sets $\left\{T^l_m(\lambda) : l-m = 0\,\mathrm{mod}(2)\right\}$ and $\left\{T^l_m(\lambda) : l-m = 1\,\mathrm{mod}(2)\right\}$ which are sets of components of irreducible tensor operators T^l and T^{l-1} for the Lie subalgebra $sl(2)$. Note that the sets $\left\{T^l_m(\lambda) : l-m = 0\,\mathrm{mod}(2)\right\}$ and $\left\{T^l_m(\lambda) : l-m = 1\,\mathrm{mod}(2)\right\}$ differ in degree because we have $\left|T\left(e^l_m(\lambda)\right)\right| = l-m+\lambda\,\mathrm{mod}(2)$. This splitting is not surprising because the components of an irreducible tensor operator have the same transformation rule as the basis vectors of the irreducible representation. On the other hand, it is known that, with respect to $sl(2)$, a graded representation space V^l of the irreducible representation of $osp(1\,|\,2)$ is a direct sum of two subspaces

$$V^l = D^l(\lambda)\oplus D^{l-1}(\lambda+1)$$

where $D^l(\lambda)$ and $D^{l-1}(\lambda+1)$ are the irreducible representation spaces of the Lie algebra $sl(2)$. Thus our general definition of tensor operators for $\mathbb{Z}_2$-graded Hopf in the case of $U_q[osp(1\,|\,2)]$, in the limit $q\to 1$ leads to the classical definition of tensor operators for the Lie algebra $sl(2)\subset osp(1\,|\,2)$.

From the Wigner–Eckart theorem it follows that it is sufficient to know one particular value of the matrix element $\left[T^l_m(\lambda)\right]_{pq}$ of tensor operator component $T^l_m(\lambda)$ to determine the reduced matrix element α and then to express all remaining matrix elements $\left[T^l_m(\lambda)\right]_{pq}$ in terms of Clebsch–Gordan coefficients. It will be applied in the next section.

4. Applications of Wigner–Eckart theorem

In this section we will consider tensor operators for the quantum superalgebra $U_q[osp(1\,|\,2)]$. First we construct in $U_q[osp(1\,|\,2)]$ irreducible representations of highest weight l (even natural number) which will be irreducible subrepresentations of the adjoint representation $(U_q[osp(1\,|\,2)], \mathrm{ad})$.

Proposition 3. *Let us define for any even natural l*

$$t^l_m = \left(\frac{[l+m]!}{[2l]![l-m]!\gamma^{l-m}}\right)^{\frac{1}{2}}\mathrm{ad}v_-^{l-m}\cdot v_+^l q^{lH}$$

where $-l \leqslant m \leqslant l$. *Then*

$$\mathrm{ad} e \cdot t^l_m = (-1)^{l-m}(([l-m][l+m+1]\gamma)^{\frac{1}{2}} t^l_{m+1} \tag{4.1}$$

$$\mathrm{ad} f \cdot t^l_m = ([l+m][l-m+1]\gamma)^{\frac{1}{2}} t^l_{m-1} \tag{4.2}$$

$$\mathrm{ad} H \cdot t^l_m = \frac{m}{2} t^l_m. \tag{4.3}$$

We have also $\left|t^l_l\right| = \lambda = l = 0 \bmod(2)$ *and* $\left|t^l_m\right| = m \bmod(2)$. *Therefore the vectors* t^l_m *form a basis of irreducible representation* (U^l, ad) *of* $U_q[osp(1\,|\,2)]$ *where* $U^l \subset U_q[osp(1\,|\,2)]$.

Proof. A direct calculation shows that t^l_l is a highest weight vector of weight $\frac{l}{2}$. Applying the standard procedure of construction of the irreducible highest weight modul of $U_q[osp(1\,|\,2)]$ gives the result. □

Corollary 1. *The elements* $t^l_m \in U_q[osp(1\,|\,2)]$ *are components of the tensor operator* $L^l \in \mathrm{Hom}_{U_q[osp(1\,|\,2)]}\left(U^l_{ad}, \mathrm{Hom}(U_q[osp(1\,|\,2)]_L, U_q[osp(1\,|\,2)]_L)\right)$.

Proof. The left regular action $L : U_q[osp(1\,|\,2)]_{ad} \to \mathrm{Hom}(U_q[osp(1\,|\,2)]_L, U_q[osp(1\,|\,2)]_L)$ is a tensor operator (examples 7, 9) and U^l is an irreducible subrepresentation of $U_q[osp(1\,|\,2)]$. So it is obvious that $L^l : U^l_{ad} \to \mathrm{Hom}(U_q[osp(1\,|\,2)]_L, U_q[osp(1\,|\,2)]_L)$ is also a tensor operator. Equations (4.1)–(4.3) show that the components t^l_m of L^l satisfy the defining equation (2.8). □

As an application of the Wigner–Eckart theorem we will calculate the matrices $\pi^j\left(t^l_m\right)_{pn} \equiv \left[t^l_m(j)\right]_{pn}$ of the basis vectors t^l_m of (U^l, ad) in the representation $(V^j(\lambda), \pi^j)$. Using the defining commutation relations for $U_q[osp(1\,|\,2)]$ one can show that t^l_m are rather complicated combinations of elements H, $v_\pm$

$$t^l_m = \left(\frac{[l+m]!}{[2l]![l-m]!}\right)^{\frac{1}{2}} \sum_k^{l-m} \sum_{p=0}^{N} (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{p(p-1)}{2}} q^{-\frac{k}{2}(l+m+1)} \frac{[l]![l-m]!}{[p]![l-p]![k-p]![l-m-k]!}$$
$$\times \gamma^p v_-^{l-m} v_+^l \frac{[4H-k+l]!}{[4H-k+l-p]!} q^{mH}. \tag{4.4}$$

where $N = \min(l, k)$ and we use a symbolic notation

$$\frac{[H+m+p]!}{[H+m]!} \equiv [H+m+p]\cdots[H+m+1].$$

So a direct calculation of $\pi^j\left(t^l_m\right)_{pn}$ using matrices $\pi^j(v_\pm)_{mn}$, $\pi^j(H)_{mn}$ seems to be difficult in a general case. However, due to the Wigner–Eckart theorem it is not necessary to do it. In fact we have

Theorem 2. *The basis vectors* t^l_m *of* (U^l, ad) *have the following matrix form in the irreducible representation* $(V^j(\lambda), \pi^j)$

$$\pi^j\left(t^l_p\right)_{mn} = \alpha[lp0, jn\lambda\,|\,jm\lambda]_q$$

where

$$\alpha = (-1)^{\frac{1}{2}l(l+1)} q^{-\frac{1}{2}l(l+1)}[l]! \left(\frac{[2j+l+1]!}{[2l]![2j-l]![2j+1]!}\gamma^l\right)^{\frac{1}{2}}$$

is a reduced matrix element of the irreducible tensor operator $\pi^j : U^l \to \mathrm{Hom}(V^j(\lambda), V^j(\lambda))$.

Proof. The representation $\pi^j : U_q[osp(1\,|\,2)] \rightarrow \mathrm{Hom}(V^j(\lambda), V^j(\lambda))$ is itself a tensor operator (examples 6, 8). Because U^l is an irreducible subrepresentation of $U_q[osp(1\,|\,2)]$ then $\pi^j : U^l \rightarrow \mathrm{Hom}(V^j(\lambda), V^j(\lambda))$ is an irreducible tensor operator. Thus according to the Wigner–Eckart theorem we have the following expression for matrix elements of components $\pi^j\left(t^l_p\right)$ of π^j

$$\pi^j\left(t^l_p\right)_{mn} = \alpha[lp0, jn\lambda\,|\,jm\lambda]_q$$

and in particular

$$\pi^j\left(t^l_l\right)_{mn} = \alpha[ll0, jn\lambda\,|\,jm\lambda]_q. \tag{4.5}$$

Now on one hand from (3.1)–(3.3) we have

$$\pi^j\left(t^l_l\right)_{mn} = (-1)^{\frac{1}{2}l(l+1)+l(j-m+l)}\left(\frac{[j-m+l]![j+m]!}{[j+m-l]![j-m]!}\gamma^l\right)^{\frac{1}{2}} q^{\frac{1}{2}l(m-l)}\delta_{mn+l}$$

and on the other we have [2]

$$[ll0, jn\lambda\,|\,jm\lambda]_q = q^{-\frac{n}{2}} q^{\frac{1}{4}(2j-l)(l+1)-\frac{1}{2}(j-m)(l+1)}$$
$$\times\left([2j+1]\frac{[2l]![2j-l]![j+m]![j-m+l]!}{[2j+l+1]![l]![l]![j-m]![j+m-l]!}\right)^{\frac{1}{2}}\delta_{mn+l}.$$

After substitution of the last two equations to equation (4.5) we get the value of α. □

At the end of this paper we give a method of constructing some elements of the centre of $U_q[osp(1\,|\,2)]$ by use of the particular C-Gc $C^l_{mn}(\lambda)$ (3.6) and the elements t^l_p of $U_q[osp(1\,|\,2)]$. It is known that $C^l_{mn}(\lambda)$ couple two irreducible representations (V^j, π^j) and (V^i, π^i) to a one-dimensional trivial representation. Therefore for any two irreducible representations (U^j, ad) and (U^i, ad) with bases $\left\{t^j_m\right\}$ and $\left\{t^i_n\right\}$, the following elements $\mathfrak{C}^j$ of $U_q[osp(1\,|\,2)]$

$$\mathfrak{C}^j = \sum_{mn}(jm\lambda_j, in\lambda_i\,|\,00)_q t^j_m t^i_n$$

form the one-dimensional trivial representation $(\mathfrak{C}^j, \varepsilon)$ described in example 5. It means that $\mathfrak{C}^j \in U_q[osp(1\,|\,2)]_\varepsilon$ and consequently, from proposition 1, belongs to the centre of $U_q[osp(1\,|\,2)]$.

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